Radiation damping effects on Dicke's maser model

1977 J. Phys. A: Math. Gen. 10 L25

(http://iopscience.iop.org/0305-4470/10/1/006)

J. Phys. A: Math. Gen., Vol. 10, No. 1, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

Radiation damping effects on Dicke's maser model

Miguel Orszag
Physics Department, Ryerson Polytechnic Institute, Toronto, Canada

Received 11 October 1976

Abstract. The critical temperature for the 'normal' to 'super-radiant' phase transition and the free energy are modified by the presence of phonons. Results show that whenever there are photons present, there are also phonons.

A remarkable property of Dicke's model for a maser (Dicke 1954) is that for certain values of the coupling constant ($\lambda > \lambda_c$), a phase transition is present (Hepp and Lieb 1973).

As far as the author knows, there is no experimental evidence for the 'normal' to 'super-radiant' phase transition.

Since Dicke's model is only valid when the wavefunctions are non-overlapping (i.e. low density radiating media) and considering that Lieb's phase transition occurs at a high coupling constant, the model does not seem suitable to describe a realistic physical system. A more realistic model is one in which a loss mechanism for the field is introduced, in the form of many-mode phonons.

One might expect the following two changes when phonons are present.

(*a*) Since only some of the atoms will emit radiation out of the system, one might expect a less stringent condition on the coupling constant between the atoms and the field for the phase transition to exist. The results presented here show that this is the case.

(*b*) Whenever there are photons present, one would expect radiationless transitions to occur, and therefore the presence of phonons (Pickles and Thompson 1974). This is also observed in the results.

Consider a system of spins coupled by the electromagnetic radiation only and assume a radiation damping mechanism through many loss oscillators or phonons, obeying the boson commutation rules. The Hamiltonian in question is:

$$H = \hbar\omega a^{\dagger}a + \tfrac{1}{2}\hbar\omega_0 \sum_{i=1}^{N} \sigma_i^z + \sum_{j=1}^{M} \hbar\Omega_j b_j^{\dagger} b_j + \sum_{i=1}^{N} \tfrac{1}{2}\hbar\omega\lambda\left(\frac{a^{\dagger}}{\sqrt{N}}\sigma_i^- + \frac{a}{\sqrt{N}}\sigma_i^+\right) + \sum_{j=1}^{M} \hbar k_j (a^{\dagger} b_j + a b_j^{\dagger}), \tag{1}$$

or, defining

$$\epsilon = \omega_0/\omega, \qquad \omega_f = \Omega_f/\omega, \qquad K_j = k_j/\omega, \tag{2}$$

the Hamiltonian can be written in the following way (per unit $\hbar\omega$):

$$H=\sum_{i=1}^{N}\left\{\frac{a^{\dagger}}{\sqrt{N}}\frac{a}{\sqrt{N}}+\frac{\epsilon}{2}\sigma_i^z+\frac{\lambda}{2}\left(\frac{a^{\dagger}}{\sqrt{N}}\sigma_i^-+\frac{a}{\sqrt{N}}\sigma_i^+\right)\right.$$
$$\left.+\sum_{j=1}^{M}\left[\omega_j\frac{b_j^{\dagger}}{\sqrt{N}}\frac{b_j}{\sqrt{N}}+K_j\left(\frac{a^{\dagger}}{\sqrt{N}}\frac{b_j}{\sqrt{N}}+\frac{a}{\sqrt{N}}\frac{b_j^{\dagger}}{\sqrt{N}}\right)\right]\right\}. \tag{3}$$

In equation (3), a and $a^{\dagger}$ are the annihilation and creation operators for the field and b_j, $b_j^{\dagger}$ are the corresponding operators for the jth phonon mode; λ is the coupling constant between the radiation field and the atoms and k_j the coupling constant between the jth mode of the oscillator bath.

A convenient representation is given by the state:

$$|S_1, S_2 \ldots S_N\rangle|n_1\rangle|n_2\rangle \ldots |n_M\rangle|n\rangle, \tag{4}$$

where $|n\rangle$ is the photon number state and $|n_j\rangle$ is the phonon number state for the jth mode, M is the number of modes. The commutation rules for the photons and phonons are:

$$[a, a^{\dagger}]=1,$$
$$[b_j, b_j^{\dagger}]=\delta_{ij}. \tag{5}$$

In thermal equilibrium, the partition function is:

$$Z=\mathrm{Tr}\,\mathrm{e}^{-\beta H}, \tag{6}$$

which in the present representation can be written as:

$$Z=\sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_N=\pm1}\sum_{n=0}^{\infty}\sum_{n_1=0}^{\infty}\cdots\sum_{n_M=0}^{\infty}\langle S_1\ldots S_N|\langle n|$$
$$\times\langle n_1|\langle n_2|\ldots\langle n_M|\,\mathrm{e}^{-\beta H}|n_M\rangle\ldots|n_1\rangle|n\rangle|S_1\ldots S_N\rangle. \tag{7}$$

Assume in the thermodynamic limit that (Wang and Hioe 1973, Orszag, to be published):

$$\left[\frac{a}{\sqrt{N}},\frac{a^{\dagger}}{\sqrt{N}}\right]=\left[\frac{b_j}{\sqrt{N}},\frac{b_j^{\dagger}}{\sqrt{N}}\right]=N^{-1}\to 0. \tag{8}$$

The various terms in the partition function can be separated in commuting products:

$$Z=\sum_{S_1,S_2\ldots S_N}\sum_{n,n_1,n_2\ldots n_M}\mathrm{e}^{-\beta n}\exp\left(-\beta\sum_{j=1}^{M}\omega_j n_j\right)\langle n_1|\ldots\langle n_M|\langle n|\exp\left[-\beta\sum_j K_j(a^{\dagger}b_j+ab_j^{\dagger})\right]$$
$$\times|n\rangle|n_1\rangle\ldots|n_M\rangle\langle S_1\ldots S_N|\Pi_{i=1}^{N}\,\mathrm{e}^{-\beta h_i}|S_1\ldots S_N\rangle, \tag{9}$$

where

$$h_i=\tfrac{1}{2}\epsilon\sigma_i^z+\tfrac{1}{2}\lambda(a\sigma_i^{+}+a^{\dagger}\sigma_i^{-}). \tag{10}$$

The diagonalization of h_i is straightforward, giving the following eigenvalues:

$$\mu_{1,2}=\pm\frac{\epsilon}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}\frac{(a^{\dagger}a)}{N}\right)^{1/2}. \tag{11}$$

The eigenvalues are actually operators and equation (11) is interpreted as a power series expansion of the $(a^\dagger a)$ operator. Since $\mu_{1,2}$ is diagonal in the $|n\rangle$ representation, we can write:

$$Z=\sum_{n_1,n_2\ldots n_M,n}\exp\left(-\beta n-\beta\sum_{j=1}^{M}\omega_j n_j\right)\left[2\cosh\left\{\frac{\epsilon\beta}{2}\left[1+\frac{4\lambda^2}{\epsilon^2}\left(\frac{n}{N}\right)\right]^{1/2}\right\}\right]^N$$
$$\times\langle n_1|\langle n_2|\ldots\langle n_M|\langle n|\exp[-\beta\,\Sigma_{j=1}^{M}K_j(a^\dagger b_j+ab_j^\dagger)]|n\rangle|n_1\rangle\ldots|n_M\rangle. \qquad (12)$$

A simple calculation proves that:

$$\langle n_1|\langle n|\exp[-\beta K_1(a^\dagger b_1+ab_1^\dagger)]|n\rangle|n_1\rangle=I_0(2\beta K_1\sqrt{nn_1}) \qquad (13)$$

where $I_0(2\beta K_1\sqrt{nn_1})$ is the zeroth order modified Bessel function.

Define the parameters:

$$x_f=n_f/N, \qquad x=n/N, \qquad (14)$$

then

$$Z=\sum_{\substack{x_1,x_2\ldots x_M,x\\(\text{steps } N^{-1})}}\exp\left[N\left(-\beta x-\beta\sum_{j=1}^{M}\omega_j x_j+\sum_{j=1}^{M}\ln(I_0(2\beta K_j N\sqrt{xx_j}))\right.\right.$$
$$\left.\left.+\ln\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]\right\}\right)\right]. \qquad (15)$$

For a large argument, $I_0(2\beta K_j N\sqrt{xx_j})$ can be approximated as:

$$I_0(2\beta K_j N\sqrt{xx_j})=[\exp(2\beta K_j N\sqrt{xx_j})](4\pi\beta K_j N\sqrt{xx_j})^{-1/2}. \qquad (16)$$

Neglecting a term of the order $(\ln N)/N$, the partition function is:

$$Z=\int_{x_1}\int_{x_2}\ldots\int_{x_M}\int_{x}\mathrm{d}x_1\ldots\mathrm{d}x_M\,\mathrm{d}x\,\exp\left[N\left(-\beta x-\beta\sum_{j=1}^{M}\omega_j x_j+2\beta\sum_{j=1}^{M}K_j\sqrt{xx_j}\right.\right.$$
$$\left.\left.+\ln\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]\right\}\right)\right]. \qquad (17)$$

This integral can be evaluated by the method of steepest descent. The result for the free energy is:

$$f=-\frac{1}{N\beta}\ln Z=\chi^*\left(1-\sum_{j=1}^{M}K_j^2/\omega_j\right)-\ln[2\cosh(\tfrac{1}{2}\epsilon\beta\eta)], \qquad (18)$$

where:

$$\eta=\left(1+\frac{4\lambda^2}{\epsilon^2}x^*\right)^{1/2} \qquad (19)$$

and x^* is solution of:

$$\tanh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]=\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\left(\frac{\epsilon}{\lambda^2}\right)\left(1-\sum_{j=1}^{M}\frac{K_j^2}{\omega_j}\right). \qquad (20)$$

The condition on the number of phonons is:

$$\sqrt{\chi_j}=\frac{K_j}{\omega_j}\sqrt{x}. \qquad (21)$$

Since

$$\tanh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]\leqslant 1,$$

equation (20) has solution $\chi \neq 0$ only if:

$$\lambda^2>\epsilon\left(1-\sum_{j=1}^{M} K_j^2/\omega_j\right). \tag{22}$$

With no coupling between the electromagnetic field and the reservoir (i.e. $K_j = 0$), the condition (22) reduces to $\lambda^2>\epsilon$, found previously by several authors.

The critical temperature, obtained by setting $x = 0$, turns out to be:

$$\tanh\left(\frac{\epsilon\beta_c}{2}\right)=\frac{\epsilon}{2\lambda^2}\left(1-\sum_{j=1}^{M}\frac{K_j^2}{\omega_j}\right). \tag{23}$$

We now summarize the results.

If

$$\lambda<\lambda_c=\sqrt{\epsilon}\left(1-\sum_{j=1}^{M} K_j^2/\omega_j\right), \tag{24}$$

there is no solution for equation (20), and the maximum exponent in the integration procedure is obtained for $x = x_j = 0$. The free energy in this case is:

$$f=-\ln[2\cosh(\tfrac{1}{2}\epsilon\beta)]. \tag{25}$$

If $\lambda>\lambda_c$ and $\beta<\beta_c$, again there is no solution for equation (20) and the free energy is given by equation (25). This is the 'normal state', characterized by no photons or phonons. If $\lambda>\lambda_c$ and $\beta>\beta_c$, equation (20) has a solution $x\neq 0$, $x_j\neq 0$. This is the 'super-radiant' state. The relation between the average (statistical) number of phonons and photons is given by equation (21).

References

Dicke R H 1954 *Phys. Rev* **93** 99
Hepp K and Lieb E H 1973 *Ann. Phys, NY* **76** 360
Pickles J B and Thompson B V 1974 *Phys. Lett.* **48A** 487
Wang Y K and Hioe F T 1973 *Phys. Rev.* A **7** 831